COMBINATORICA 4 (1) (1984) 1—6

ON AN OBSTRUCTION FOR PERFECT MATCHINGS

Ron AHARONI

Dedicated to Paul Erdős on his seventieth birthday

Received 5 January 1983
Revised 20 May 1983

Steffens [3] introduced a substructure (called below a "compressed set") which prevents a graph from having a perfect matching, and proved that a countable graph possesses a perfect matching if and only if it does not contain such a substructure. In this paper we study some properties of compressed sets.

1. Introduction

In [3] Steffens proved a necessary and sufficient condition for the existence of perfect matchings in countable graphs. He presented a substructure (called below a "compressed set") which obstructs perfect matchings in any graph, and proved that in the countable case this is the only possible obstruction for perfect matchings. It appears that this obstruction also plays a vital role in the characterization of graphs possessing perfect matchings in the case of higher cardinalities: in [2] a characterization involving the concept of compressed sets is proved for graphs of size $\aleph_1$, and it is conjectured that a similar result holds for general graphs*. It is therefore worthwhile studying the properties of compressed sets, and this is the main purpose of the present paper. Among other properties we show that (a) deleting edges from an obstructed graph results in an obstructed graph; (b) given a graph $G=(V,E)$ and a subset S of V, if neither of the subgraphs of G spanned by S and by $V\setminus S$ contain a compressed set then G itself does not contain a compressed set; and that (c) Steffens' criterion holds also for graphs all but countably many of whose vertices are covered by a matching.

2. Definitions and notation

All graphs in this paper are undirected. A graph whose vertex set is U and whose edge set is I will be denoted by (U, I). The symbol G will always denote a graph (V, E), and unless otherwise stated we shall be referring to a graph with which these symbols are associated. We shall write $A\dot{\times}B$ for the set of *unordered* pairs $\{\{a,b\}: a\in A,\ b\in B\}$. If $S\subset V$ then $G[S]$ denotes the graph $(S, E\cap(S\dot{\times}S))$ and

AMS subject classification (1980): 04 A 20, 03 E 05

* This conjecture has now been proved.

$G-S$ denotes $G[V \setminus S]$. If F is a subset of E, A a subset of V and a an element of V, then $F\langle a\rangle$ denotes $\{b\in V: [a, b]\in F\}$, $F[A]$ denotes $\bigcup\{F\langle a\rangle: a\in A\}$, and $F(a)$ denotes the element of $F\langle a\rangle$ if $|F\langle a\rangle|=1$. The *restriction* of F to A, $F\restriction A$, is defined as $F\cap(A\dot{\times}V)$. The *support* of F, $s(F)$, is defined as $F[V]$. A subset F of E is a *matching* if $|F\langle a\rangle|\leqq 1$ for every $a\in V$ (i.e. no two edges in F are incident) and is a *perfect matching* if $|F\langle a\rangle|=1$ for every $a\in V$ (in other words, if it is a matching and its support is V). If F is a matching and $S\subset V$ then F is said to be a *matching of* S if $F\subset S\dot{\times}V$ and $s(F)\supset S$. A subset S of V is *matchable* if it has a matching, and is *tight* if it is matchable and $s(F)=S$ for every matching F of S. (This corresponds to the notion of an "independent subgraph" in [3].) A graph is *loose* if it contains no non-empty tight set. A subset C of V is *compressed* if $C=T\cup\{z\}$ for some $T\subset V$ and $z\in V\setminus T$ such that T is tight and $E\langle z\rangle\subset T$. We then write $C=c(z, T)$. If G contains a compressed set then it is said to be *obstructed.*

A *path* P in G is a sequence $(a_k: \alpha<k<\beta)$ of vertices, where $-\omega\leqq\alpha\leqq\beta\leqq\omega$ and $[a_k, a_{k+1}]\in E$ for each k. The set $\{[a_k, a_{k+1}]: \alpha<k$ and $k+1<\beta\}$ of edges of P is denoted by $\mathrm{ED}(P)$ and the set $\{a_k: \alpha<k<\beta\}$ by $\mathrm{VR}(P)$. The path $(a_{-k}: \alpha<<k<\beta)$ is denoted by $\tilde{P}$. If $y=a_j$ for some $\alpha<j<\beta$ we write $Py=(a_k: \alpha<k\leqq j)$ and $yP=(a_k: j\leqq k<\beta)$. If P and Q are paths sharing only one common vertex y which is the last vertex on P and the first vertex on Q, we write $P*Q$ for the path R satisfying $Ry=P$ and $yR=Q$. If F is a matching in G then a path P is called *F-alternating* if precisely one of each two adjacent edges in P belongs to F. If F is a matching and $z\in V$ then $l(z, F)$ denotes the set of vertices x for which there exists an F-alternating path from z to x, whose first edge does not belong to F and whose last edge belongs to F. We also write $n(z, F)=F[l(z, F)]$ and $m(z, F)==l(z, F)\cup n(z, F)$.

The symmetric difference $(A\setminus B)\cup(B\setminus A)$ of two sets A and B is denoted by $A\,\Delta\,B$.

3. Preliminary lemmas

Lemma 1. *Let ζ be an ordinal and $\{T_\alpha: \alpha<\zeta\}$ an ascending continuous sequence of tight subsets of V. Then $T=\bigcup\{T_\alpha: \alpha<\zeta\}$ is tight.* ∎

For a proof see the proof of [3, Lemma 3].

Corollary 1a [3, Lemma 3]. *In any graph there exists a maximal tight set.* ∎

The proofs of the next three lemmas and their corollaries are easy and left to the reader:

Lemma 2. *If T is tight in G and S is tight in $G-T$ then $T\cup S$ is tight in G.* ∎

Corollary 2a. *If T is a maximal tight set in G then $G-T$ is loose.* ∎

Lemma 3. *If T is tight in G and C is compressed in $G-T$ then $T\cup C$ is compressed in G.* ∎

Corollary 3a. *If G is unobstructed and T is tight in G then $G-T$ is unobstructed.* ∎

Lemma 4. *A compressed set is unmatchable.* ∎

Lemma 5. *Let F be a matching and $z \in V \setminus s(F)$. If there exists an F-alternating path P starting at z such that either:*

(a) *P terminates at a vertex $x \notin s(F) \cup \{z\}$*

or

(b) *P is infinite*

then there exists a matching of $s(F) \cup \{z\}$; if $|\mathrm{ED}(P)| > 1$ then this is also a matching of $s(F)$.

Proof. This is demonstrated by the matching $F \Delta \mathrm{ED}(P)$. ∎

Note that the existence of a path P satisfying (a) is equivalent to the condition $E[l(z, F)] \not\subset s(F) \cup \{z\}$.

Lemma 6. *If F is a matching and $z \in V \setminus s(F)$ then $m(z, F)$ is compressed if and only if there is no path satisfying conditions* (a) *or* (b) *in Lemma 5.*

Proof. If there is a path satisfying conditions (a) or (b) then, by Lemma 5, $m(z, F)$ is matchable and therefore not compressed.

Assume now that there is no path satisfying (a) or (b). Let $K = F \restriction m(z, F)$ and $T = s(K)$. Then $E\langle z \rangle \subset T$, or else a path satisfying (a) would exist. It remains to show that T is tight. Let H be a matching of T and suppose that there exists a vertex $x \in s(H) \setminus T$. By (a), $y = H(x) \notin l(z, F)$, and hence $u = K(y) \in l(z, F)$. The vertex x is the starting point of a path P which is a connected component in the graph $(V, H \cup K)$. By (b) P is finite, and thus has a last vertex v. Since $s(H) \supset s(K)$ there holds $v \in s(H) \setminus s(K)$. Let Q be an F-alternating path from z to u and let r be the first vertex on Q belonging to $\mathrm{VR}(P)$. If P and Q traverse the edge $[r, K(r)]$ in the same direction then the path $Qr * rP$ satisfies (a). If P and Q traverse $[r, K(r)]$ in opposite directions then $Qr * r\overleftarrow{P}$ satisfies (a). ∎

From Lemmas 5 and 6 there follows:

Corollary 6a. *If $C = c(z, T)$ is compressed and F is a matching of T then $m(z, F)$ is compressed.* ∎

Lemma 7. *Let x be a vertex of $G, C = c(z, T)$ a compressed set in $G - \{x\}$ and F a matching of T in $G - \{x\}$. Then precisely one of the following two possibilities holds: $m(z, F)$ is compressed in G, or $B = C \cup \{x\}$ is tight in G.*

Proof. If B is tight then it is matchable, and hence so is its subset $m(z, F)$, and thus $m(z, F)$ cannot be compressed. Assume that $m(z, F)$ is not compressed. By Lemma 6 it follows that there exists an F-alternating path P from z to x. The matching $F \Delta \mathrm{ED}(P)$ then shows that B is matchable. Let H be a matching of B. Denote by L the graph $(V, F \cup H)$. The vertex z is a starting point of a path Q which is a connected component of L. By Lemma 6, Q must terminate at x. The matching $H' = H \Delta \mathrm{ED}(Q)$ is then a matching of T in $G - \{x\}$, and hence $s(H') = T$. But $s(H) = s(H') \cup \{z, x\}$, and therefore $s(H) = B$. ∎

Corollary 7a [3, Lemma 6]. *If G is unobstructed and loose then $G - \{x\}$ is unobstructed for any $x \in V$.* ∎

Lemma 8. *Let T and S be tight sets, and let F be a matching of T and H a matching of S. If $z \in H[T] \setminus T$ then $T \cup (S \setminus H[T]) \cup \{z\}$ is compressed.*

Proof. Denote $R=T\cup(S\setminus H[T])$. Then $F\cup(H\setminus H\restriction T)$ is a matching of R, and so R is matchable. Let us show that R is tight. Let J be a matching of R, and $J'=J\restriction T$. Suppose that $u\in s(J)\setminus R$ for some $u\in V$. Let K be the graph $(V, J\cup H)$. Then u is the starting point of a path P in K. By Lemma 5(b), P is finite, so it contains a last vertex p. Since the degree of p in K is 1, either (i) $p\in s(H)\setminus s(J)$ or (ii) $p\in s(J)\setminus s(H)$. Assume that (i) holds. Then $p\in s(H\restriction T)$. Since $u\notin T$ there exists a first vertex w on P (starting from p) such that $w\neq p$ and $w\notin T$. Since T is tight $J[T]=T$ and hence $w\in H[T]\setminus T$. Let $r=J(w)$. Then $r\in l(p, J')$ and $w\in E\langle r\rangle\setminus T$, which, by Lemma 5(a), contradicts the tightness of T. Assume now that (ii) holds. Then $p\notin S$ and $u\in E[l(p, H)]$, contradicting the tightness of S, by Lemma 5(a).

To complete the proof of the lemma it suffices to show that $E\langle z\rangle\subset R$. Suppose that $y\in E\langle z\rangle\setminus R$ for some $y\in V$. Let L be the graph $(V, F\cup H)$. The vertex z is the starting point of a path Q in L. Since T is tight it follows that Q is finite, and hence it has a last vertex q. By Lemma 5(a), $q\in T$, and since the degree of q in D is 1 we have $q\in T\setminus S$. Clearly, then, $z\in l(q, H)$, and since S is tight, Lemma 5(a) yields that $y\in S$. Since $y\notin R$, it follows that $H(y)\in T$. The vertex y is the starting point of a path Q' in L. By Lemma 5(b) Q' is finite, so it has a last vertex q'. Since T is tight it follows from Lemma 5(a) that $q'\in T\setminus S$. But this clearly implies that $q'\in E[l(q, H)]$, which, by Lemma 5, contradicts the tightness of S. ▮

Corollary 8a. *If T and S are tight then either $T\cup S$ is tight or $T\cup S$ contains a compressed set.*

Proof. Let F and H be matchings of T and S, respectively. If $H[T]\setminus T\neq\emptyset$ then, by the lemma, $T\cup S$ contains a compressed set. On the other hand, if $H[T]\setminus T=\emptyset$ then $s\big(H\restriction(V\setminus T)\big)\subset V\setminus T=V\setminus s(F)$, and hence $F\cup\big(H\restriction(V\setminus T)\big)$ is a mathcing of $T\cup S$, so $T\cup S$ is matchable. If I is a matching of $T\cup S$ then $s(I)=s\big(I\restriction(T\cup S)\big)=s(I\restriction T)\cup s(I\restriction S)=T\cup S$, and hence $T\cup S$ is tight. ▮

Lemma 9. *If S is a tight subset of V and $G-S$ is unobstructed then G is unobstructed.*

Proof. Let F be a matching of S. Suppose that G contains a compressed set $C=c(z, T)$. Let H be a matching of T and let L be the graph $(V, F\cup H)$. Let $I=H\setminus H\restriction S$.

Consider first the case $z\in S$. The vertex z is the starting point of a path P which is a connected component in L. Since T is tight P is finite and hence ends at some vertex $x\in T\setminus S$. We whall reach a contradiction by showing that $m(x, I)$ is compressed in $G-S$. Suppose that this is not the case. By Lemma 6 there exists then an I-alternating path Q which either (a) terminates at a vertex $u\in V\setminus S\setminus s(I)$ or (b) is infinite. If (b) occurs then $P*Q$ contradicts the assumption that T is tight. If (a) occurs, then, since T is tight, $u\in T$ and hence $u\in H[S]\setminus S$. Let R be the path starting at u which is a connected component of L. Since S is tight R is finite and thus ends at some vertex $v\in V\setminus T$. Since $u\notin \mathrm{VR}(P)$ and both P and R are connected components in L it follows that $\mathrm{VR}(P)\cap\mathrm{VR}(R)=\emptyset$. The path $P*Q*R$ contradicts then, by Lemma 5(a), the tightness of T.

Assume now that $z\notin S$. We show that $m(z, I)$ is compressed in $G-S$. If not then there exists an I-alternating path Q starting at z which is either infinite or ends at a vertex $v\in V\setminus X\setminus s(I)$. Since T is tight $v\in H[S]\setminus S$. Let R be the path starting at v which is a connected component in L. Then, like in the case $z\in S$, the path $Q*R$ contradicts the tightness of T. ▮

Some basic properties of compressed sets and unobstructed graphs

Lemma 10. *If G is loose and unobstructed and $a \in V$ then $G-\{a, x\}$ is unobstructed for some $x \in E\langle a\rangle$.*

Proof. We have $E\langle a\rangle \neq \emptyset$, for otherwise the set $\{a\}=c(a, \emptyset)$ is compressed. Let $\{x_\alpha: \alpha<\lambda\}$ be an enumeration of $E\langle a\rangle$, where $\lambda=|E\langle a\rangle|$. If the lemma is false, then for every $\alpha<\lambda$ there exists a subset C_α of $V \setminus \{a\}$ which is compressed in $G-\{a, x_\alpha\}$. Denote $G'=G-\{a\}$, and for each $\alpha<\lambda$ let $C'_\alpha=C_\alpha \cup \{x_\alpha\}$. We show that C'_α is tight in G' for each $\alpha<\lambda$. If this is false then, by Lemma 7, G' contains a compressed set. By Lemma 7 again this yields that G contains either a compressed set or a nonempty tight set, contrary to the lemma's hypothesis.

For each $\alpha \leqq \lambda$ let $T_\alpha=\bigcup\{C'_\beta: \beta<\alpha\}$.

Assertion: T_α is tight in G' for every $\alpha \leqq \lambda$.

Proof of the assertion: Clearly $T_0=\emptyset$ is tight. If β is a limit ordinal and the assertion has been shown for all $\alpha<\beta$ then the assertion for $\alpha=\beta$ follows by Lemma 1. Assume now that the assertion holds for $\alpha=\beta$, and let us show it for $\alpha=\beta+1$. If $T_{\beta+1}$ is not tight in G' then, since $T_{\beta+1}=T_\beta \cup C'_\beta$ and both T_β and C'_β are tight in G', it follows by Cor. 8a that G' contains a compressed set. By Lemma 7 this implies that G contains either a compressed set or a nonempty tight set, contradicting the hypothesis of the lemma. The assertion is thus proved.

Write: $T=T_\lambda$. By the assertion, T is tight in G'. Let us show that T is tight in G as well. If not, then there exists a matching H of T such that $s(H) \supsetneqq T$. If $a \notin s(H)$ then H is a matching of T in G', contrary to the fact that T is tight in G'. Therefore $a \in s(H)$, which means that $H(a)=x_\alpha$ for some $\alpha<\lambda$. Then $H \setminus H \upharpoonright \{x_\alpha\}$ is a matching of $T \setminus \{x_\alpha\}$ in $G-\{a, x_\alpha\}$. But $C_\alpha \subset T \setminus \{x_\alpha\}$, and by Lemma 4 it is unmatchable in $G-\{a, x_\alpha\}$, a contradiction. Thus T is tight in G, and since $E\langle a\rangle \subset T$ it follows that $T \cup \{a\}$ is compressed, contrary to the assumption that G is unobstructed. ▮

We next show that the condition in Lemma 10 that G is loose is redundant:

Theorem 1. *If G is unobstructed and $a \in V$ then $G-\{a, x\}$ is unobstructed for some $x \in E\langle a\rangle$.*

Proof. Let T be a maximal tight set in G and let F be a matching of T. If $a \in T$ then let $x=F(a)$. Denoting $G'=G-\{a, x\}$, $T'=T \setminus \{a, x\}$ is clearly tight in G', $G'-T'=G-T$ is unobstructed by Cor.3a, and hence, using Lemma 9, G' is unobstructed. Assume now that $a \notin T$. By Cors. 2a and 3a, $G-T$ is loose and unobstructed, and by Lemma 10 it follows then that $G-T-\{a, x\}$ is unobstructed for some $x \in E\langle a\rangle \setminus T$. Denoting $G'=G-\{a, x\}$, $G'-T=G-T-\{a, x\}$ is unobstructed and T is tight in G', and hence, by Lemma 9, G' is unobstructed. ▮

Theorem 1 is an important tool in studying unobstructed graphs. As a first application of it we derive a strengthening of Steffens' result.

Theorem 2. *If G is unobstructed and $G-s(F)$ is countable for some matching F then G possesses a perfect matching.*

Proof. Denote $A=V \setminus s(F)$, and let $\{a_i: i<|A|\}$ be an enumeration of A. We construct inductively sequences of distinct vertices c_k and $y_k=H(c_k)$, so that $G_k=G-\bigcup\{\{c_j, y_j\}: j<k\}=(V_k, E_k)$ is unobstructed for each k. Suppose that c_j

and y_j have been defined for all $j<k$. We let c_k be a_i for the first i such that $a_i \in V_k$, unless 1) $A \cap V_k = \emptyset$ or 2) k is odd and $\{F(y_j): j<k$ and $y_j \in s(F)\} \cap V_k \neq \emptyset$, in the latter case we choose c_k to be $F(y_j)$ for the first j such that $y_j \in s(F)$ and $F(y_j) \in \in V_k$. In any of these cases we use Theorem 1 to choose an element y_k of $E_k\langle c_k \rangle$ such that $G_{k+1} = G - \{c_k, y_k\}$ is unobstructed. The construction ends after ω steps or when $(A \cup \{F(y_j): j<k$ and $y_j \in s(F)\}) \cap V_k = \emptyset$ at some step k. By our construction H is a matching which satisfies $F[s(F) \setminus s(H)] \subset s(F) \setminus s(H)$ and $s(H) \cup s(F) = V$. Hence $H \cup (F \restriction (s(F) \setminus s(H))$ is a perfect matching of G. ∎

The next property which we consider is not unexpected, but seems to be surprisingly difficult to prove (compare with [1, Lemma 11]).

Theorem 3. *If G is obstructed and $F \subset E$ then $L = (V, F)$ is obstructed.*

Proof. Let $C = c(z, T)$ be a compressed set in G, and let M be a matching of T. Suppose that L is unobstructed. Let $v_0 = z$. By Theorem 1 there exists $u_1 \in F\langle v_0 \rangle$ such that $L_1 = L - \{v_0, u_1\} = (V_1, F_1)$ is unobstructed. Since $F\langle v_0 \rangle \subseteq E\langle v_0 \rangle \subseteq T$ it follows that $u_1 \in T$. Let $v_1 = M(u_1)$. By Theorem 1 again there exists $u_2 \in F_1\langle v_1 \rangle$ such that $L_1 - \{v_1, u_2\}$ is unobstructed. Since $v_1 \in I(z, M)$ it follows by Lemma 5(a) that $F_1\langle v_1 \rangle \subseteq T$ and thus $u_1 \in T$. Define $v_2 = M(u_2)$. In this way we construct sequences (v_k) and (u_k) of distinct vertices such that $u_k \in E\langle v_{k-1} \rangle$ and $v_k = M(u_k)$ for each $k \geqq 1$. This produces an infinite M-alternating path starting at z, contradicting, by Lemma 5(b), the tightness of T. ∎

From Theorem 3 we can deduce another property which one would naturally expect, and which is a strengthening of Lemma 9.

Theorem 4. *If $S \subset V$ and both $G[S]$ and $G - S$ are unobstructed then G is unobstructed.*

Proof. Let L be the graph $(V, E \setminus (S \dot{\times} (V \setminus S)))$. By Theorem 3 it suffices to show that L is unobstructed. Suppose that there exists a compressed set $C = c(z, T)$ in L, and let F be a matching of T. Without loss of generality we may assume that $z \in S$. Let $H = F \cap (S \dot{\times} S)$. By Cor.6a $m(z, F)$ (taken in L) is compressed in L. Clearly, $m(z, H)$ taken in $G[S]$ is equal to $m(z, F)$ taken in L, and thus $m(z, H)$ is compressed in L, which clearly implies that $m(z, H)$ is compressed in $G[S]$. This contradicts the assumption that $G[S]$ is unobstructed. ∎

References

[1] R. Aharoni, On the equivalence of two conditions for the existence of transversals, *J. Comb. Th. Ser. A.*, **34** (1983), 202—214.
[2] R. Aharoni, Matchings in graphs of size $\aleph_1$, submitted for publication.
[3] K. Steffens, Matchings in countable graphs, *Can. J. Math.* **29** (1976), 165—168.

Ron Aharoni

Department of Mathematics
Technion, Israel Institute of Technology
Technion City, 32000 Haifa, Israel